Modified Cohen's Q for testing simultaneously dependent and overlapping correlations

Lourens Waldorp Jelte Wicherts

University of Amsterdam, Roetersstraat 15, 1018 WB, the Netherlands

Abstract

The objective is to obtain a version of Cohen's Q for Fisher transformed correlations between dependent and overlapping data sets. This well-known problem of testing dependent and overlapping correlations has been addressed for pairwise comparisons. In this paper we devise a test that provides a single statistic for a user provided contrast. For example, the difference between several pairs of correlations can be tested in a single statistic. Or, subjects being measured in two conditions giving rise to two correlation matrices for which differences are to be testefd. Simulations show that the test has good finite sample behavior for larger samples ($n = 50$ or larger) and reasonable for small samples ($n = 30$).

Key words: pairwise comparisons, omnibus test on correlations, Wald test

1 Introduction

In meta analysis of correlations it is often useful to test whether the correlations from several studies differ in general or are different pairwise. Often Cohen's Q is used for Fisher transformed independent correlations. However, when the correlations are dependent and overlap (i.e., the correlations refer to an overlapping set of variables), then Cohen's Q cannot be used since the data used to obtain the correlations are dependent. This situation arises naturally. For instance, in a single study with several subtests, the subjects are the same and the correlations overlap [1]. As another example, correlation matrices of brain regions obtained from functional magnetic resonance imaging (fMRI) for the same subjects in different conditions, result in dependent correlations across conditions. In [1] several methods for pairwise comparisons between dependent and overlapping correlations are evaluated. However, there is no

Email addresses: `waldorp@uva.nl` (Lourens Waldorp), `j.wicherts@uva.nl` (Jelte Wicherts).

general method to replace Cohen's Q to obtain a single statistic which tests several or all differences between dependent correlations simultaneously. In this paper we present such a method.

Our approach is based on the limiting distribution of the sample correlation matrix [2,3]. Then, using the delta method, we obtain the limiting distribution of the Fisher transformed correlations. We then define a chi-square test to replace for Cohen's Q which takes the dependencies between the different correlations into account. In this chi-square test we can define contrasts that are of interest, including all pairwise differences between the correlations. The finite sample behavior is seen to be good for medium to large samples ($n = 50$ and larger), and reasonable for small samples ($n = 30$).

2 Method

The objective is to obtain a χ^2 test based on a contrast of the Z-transformed correlations. We use the limiting distribution result of the sample correlation matrix from Neudecker [2] based on normal data. Then we use the delta method to obtain the distribution of the Z-transformed data.

Let $X_i = (X_{i1}, \ldots, X_{in})'$ denote the random variable of variable i ($i = 1, \ldots, k$) containing the responses of n observations (e.g., subjects). For each of the k variables the observations (X_i) are collected in the $n \times k$ matrix X. We assume that X is multivariate normal with mean $1_n\mu'$, where 1_n is a vector of length n with ones, and variance matrix $I_n \otimes \Sigma$. So, the n observations on the k variables are independent.

Correlations between variables can be computed, resulting in a $k \times k$ matrix $R = S_d^{-1/2} S S_d^{-1/2}$, where $S = (n-1)^{-1} X'(I_n - n^{-1} 1_n 1_n')X$, $S_d^{-1/2} = S^{-1/2} \odot I_k$, and $S^{-1/2} = U\Lambda^{-1/2}U'$ is the inverse of the square root obtained with the eigenvalue decomposition with eigenvalues $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_k)$. Since the correlation matrix contains ones on the diagonal and is symmetric, it is convenient to use the unique elements of R. This is achieved by stacking columns of R, $\mathrm{vec}(R)$, and multiplying this with a $k(k-1)/2 \times k^2$ transition matrix D' such that $D'\mathrm{vec}(R) = r \in \mathbb{R}^q$ ($q = k(k-1)/2$) contains only the unique elements of R (excluding the diagonal) (see e.g., [3]).

Because confidence intervals based on Fisher's Z-transform are variance stabilizing [4], we use the transformed correlations. Fisher's Z-transform of the true correlation ρ is defined as

$$z : \rho \mapsto \frac{1}{2} \ln \frac{1-\rho}{1+\rho} \tag{1}$$

So we have a random $q = k(k-1)/2$ vector z containing the unique and Z-transformed correlations. In order to devise a test we need the limiting distribution of Z.

Let P denote the population correlation matrix. Assuming the data X are multivariate normal, Neudecker and Wesselman [2] showed that

$$\begin{aligned}
&\sqrt{n}(r-\rho) \to_d N_q(0,\Phi) \\
&\qquad\qquad \Phi = 2D'(I_{k^2} - N(I_k \otimes P))(P \otimes P)((I_{k^2} - (I_k \otimes P)N)D \in \mathbb{R}^q_q
\end{aligned} \tag{2}$$

where $N = K_k \odot I_{k^2}$. We use this result to obtain the limiting distribution of the Z-transformed variable z from the delta method. Let $\zeta = z(\rho)$ be the true value of the transformed correlations. As a corollary to (2) we have that

$$\sqrt{n}(z-\zeta) \to_d N_q(0, \partial_\rho z \Phi(\partial_\rho z)') \tag{3}$$

where $\partial_\rho z$ is the $q \times q$ matrix with first order partial derivatives of Z, which are $(1-\rho^2)^{-1}$ for each element. Let $\Omega = \partial_\rho z \Phi(\partial_\rho z)'$. Now that we have the limiting distribution of z, we can create a contrast C such that the hypothesis $H_0 : Cz = u$ can be tested. For any contrast C with rank $r(C\Sigma) = m$ we can define a χ^2-test

$$W = n(Cz-u)'(C\Omega C')^+(Cz-u) \tag{4}$$

where A^+ is the Moore-Penrose inverse of A. The random variable W is $\chi^2_m(\delta)$ distributed with $\delta = ||Cz_\rho - u||^2_\Omega/2$. We use this generalized inverse because the rank m of C may be smaller that the number of contrasts (rows) devised in C.

Example 1. An example of an interesting contrast is the difference between each of the correlations and their mean, that is $z - \bar{z}1_q$. Then $C = I_q - q^{-1}1_q1_q'$, such that $Cz = z - \bar{z}1_q$. For instance, the contrast with $q = 3$ is

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

And so because this contrast is idempotent we have

$$W_{\bar{z}} = n(z-\bar{z}1_q)'(C\Omega^{-1}C')^{-1}(z-\bar{z}1_q) \tag{5}$$

This is analogous to Cohen's Q but now the elements in z are correlated because there are the same subjects for each correlation, and the correlations are overlapping because the subtests are the same between different correlations. As a consequence of using this contrast, the rank of $C\Sigma$ is $r(C\Sigma) = q-1$. And so, under H_0 this test has a $\chi^2_{q-1}(0)$ distribution.

Example 2. As a second example, consider the hypothesis that the difference of correlations is zerobetween any relevant pair of variables under H_0. That means that the $(q-1)\times q$ contrast matrix is $C = I_{q-1} \otimes (1, -1)$. For example, with $q = 3$ correlations we have that

$$C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix},$$

such that

$$W_{\mathrm{p}} = n(z_1 - z_2, z_2 - z_3) \begin{pmatrix} \omega_{11} + \omega_{22} - 2\omega_{12} & \omega_{12} - \omega_{13} - \omega_{22} + \omega_{23} \\ \omega_{12} - \omega_{13} - \omega_{22} + \omega_{23} & \omega_{22} + \omega_{33} - 2\omega_{23} \end{pmatrix}^{-1} \begin{pmatrix} z_1 - z_2 \\ z_2 - z_3 \end{pmatrix}$$

The rank $r(C\Sigma) = q - 1$, so again the degrees of freedom will be $q - 1$.

Example 3. The proposed method can easily be extended to testing between two correlation matrices obtained from two (or more) different conditions from the same subjects, i.e., the data from the two conditions are dependent. Suppose that $X_1 \in \mathbb{R}_k^n$ is obtained in the first and $X_2 \in \mathbb{R}_k^n$ is obtained in the second condition, for the same n subjects, and so $X = (X_1, X_2) \in \mathbb{R}_{2k}^n$. This can be considered as a repeated measure with $2k$ variables. Then the sample covariance matrix is

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{pmatrix},$$

where $S_{12} = (n-1)^{-1} X_1 (I_n - n^{-1} 1_n 1_n') X_2$. The correlation matrix R can be partitioned similarly. The interest is in testing the unique correlations of R_{11} and R_{22}. All $q = k(2k-1)$ unique correlations can be put in z and the first and last $k(k-1)/2$ canbe compared in a simple linear contrast. The the $2k \times 2k$ covariance matrix matrix Φ of z contains the dependencies between the two conditions, similar to general linear model hypothesis testing (MANOVA) [5]. A contrast can be set up to test for the $k(k-1)/2$ unique correlation differences of interest in R_{11} and R_{22}. In general, this contrast contains $k(k-1)/2$ rows and q columns with each row $(1, 0_{2(2k)}, -1, 0, \ldots, 0)$. For $k = 3$ the vector z contains $q = k(2k-1) = 15$ correlations, of which only 3 differences are important to consider $(R_{11} - R_{22})_{21}$, $(R_{11} - R_{22})_{31}$, and $(R_{11} - R_{22})_{23}$. The contrast matrix to obtain these differences rom z is

$$C = \begin{pmatrix} 1, 0_{12}, -1, 0, 0 \\ 0, 1, 0_{12}, -1, 0 \\ 0, 0, 1, 0_{12}, -1 \end{pmatrix}.$$

The rank $r(C\Sigma) = k(k-1)/2$, which defines the degrees of freedom.

3 Numerial example

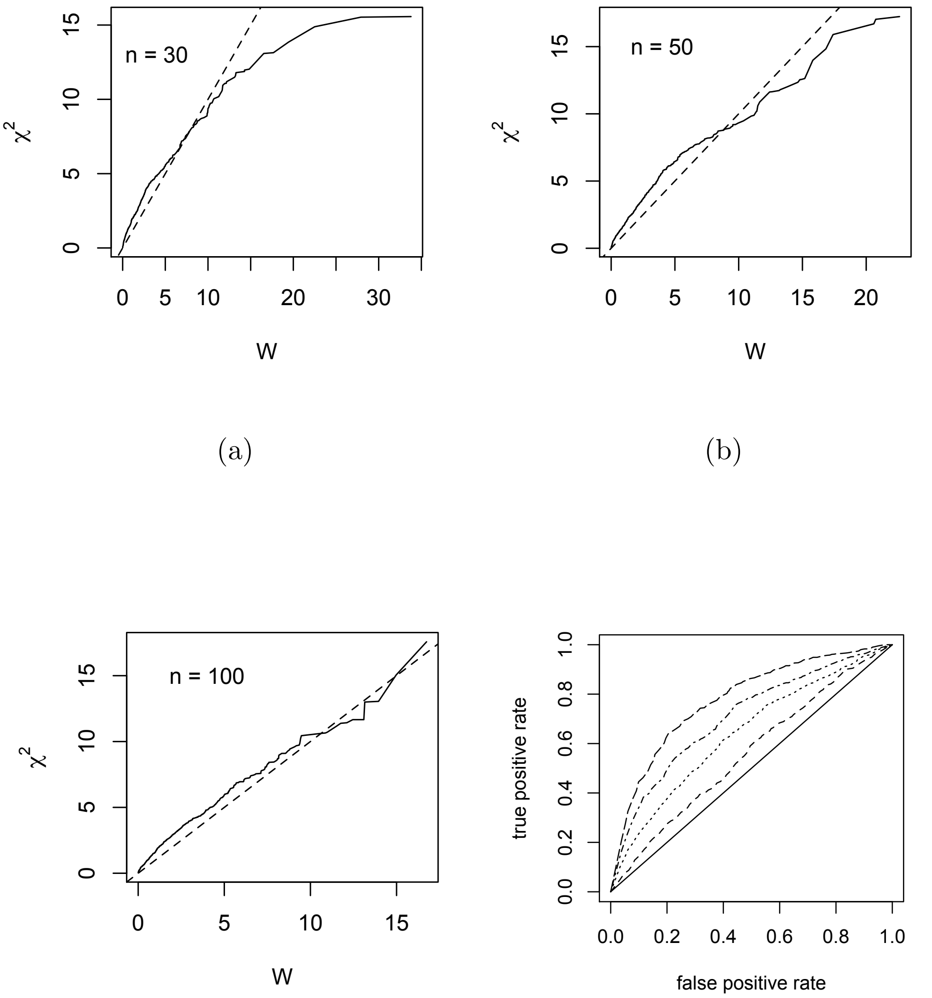
To show finite sample behavior of W we provide a numerical example. In this example we use the contrast that tests between the transformed correlations and its average for $k = 3$ variables and $n = 50$, and 100 subjects. This gives $q = 3$ unique correlations. The contrast is $Cz = z - \bar{z}1_3$, as in *Example 1* above and so, there are 2 degrees of freedom. We generate 1000 times a 3×3 correlation matrix R from normally distributed data such that all three unique correlations are equal to 0.5. This is done by generating each variable $X_i = U + Z$ $(i = 1, 2, 3)$, where $U \sim N(0, 1)$ and $Z \sim N(0, 1)$ are independent. Then the null hypothesis $H_0 : z = \bar{z}1_3$ is true. We compare the quantiles of $W_{\bar{z}}$ to the $\chi_2^2(0)$ distribution. It can be seen in Figure 1(a)-(c) that the quantiles of $W_{\bar{z}}$ are very close to the theoretical quantiles of the $\chi_2^2(0)$ distribution for $n = 100$. However, in the tails (large quantiles), the quantiles of $W_{\bar{z}}$ are overestimated slightly when $n = 50$ and more when $n = 30$.

To investigate the false positive rate (FPR) and true positive rate (TPR), that is, Type I error and power, respectively, we use receiver operating characteristic (ROC) curves. In an ROC curve the FPR and FPR are plotted to indicate whether the FPR remains low while the TPR increases as the threshold (significance level) is increased [6]. For the FPR we use several values of effect size $||\beta - \bar{\beta}|| = 0$, 0.408, 1.225, 1.466, and 2.268, for the coefficients $X_i = \beta U + Z$; the coefficients are respectively $\beta = (0.5, 0.5, 0.5)'$, $(0.5, 0.5, 1)'$, $(0.5, 0.5, 2)'$, $(0.01, 0.5, 2)'$, and $(0.01, 0.5, 3)'$. We used $n = 30$ and 1000 replications for each point in the ROC curve. Figure 1(d) shows that for a small sample, $n = 30$, the FPR remains low for reasonable values of effect size while the power is sufficient to be able to detect differences. For example, with only $n = 30$ observations and a medium effect size of 1.466 and an FPR of 0.2, the TPR is about 0.6.

4 Discussion

When testing differences between correlations with overlapping variables or obtained from the same set of subjects, Cohens Q is inappropriate. Methods to cope with dependencies between correlations appear to exist for pairwise comparisons only. We devised a method to test simultaneous comparisons of differences between correlations. The chi-square test we developed has good properties for large samples ($n = 50$ or larger) and reasonable properties for small samples ($n = 30$).

The proposed method can also be extended to testing partial correlations. This is an important subject, especially with the rise of graphical models [7]. In a simple graph (i.e., without self-loops and multiple connections) a zero partial correlation indicates that two variables are independent given the set of remaining variables. Graphical models have applications in the analysis of social and psychological data [8] as well as neuroscience [9].



(a) (b)

(c) (d)

Fig. 1. Finite sample behavior is shown of $W_{\bar{z}}$ for $k = 3$ subtests and subjects $n = 30$, 50, and 100. The empirical distribution of $W_{\bar{z}}$ is compared with the theoretical $\chi_2^2(0)$ distribution in qq-plots in (a), (b), and (c). In (d) ROC curves are shown for $n = 30$ and effect size 0 (solid), 0.408 (dashed), 1.225 (dotted), 1.466 (dashed-dotted), and 2.266 (long dash).

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